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Mathematical foundations of the infinity computer

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Abstract

All the existing computers are able to do arithmetical operations only with finite numerals. Operations with infinite and infinitesimal quantities could not be realized. The paper describes a new positional system with infinite radix allowing us to write down finite, infinite, and infinitesimal numbers as particular cases of a unique framework. The new approach both gives possibilities to do calculations of a new type and simplifies fields of mathematics where usage of infinity and/or infinitesimals is required. Usage of the numeral system described in the paper gives possibility to introduce a new type of computer – Infinity Computer – able to operate not only with finite numbers but also with infinite and infinitesimal ones.

1. Introduction

Problems related to the idea of infinity are among the most fundamental and have attracted the attention of the most brilliant thinkers throughout the whole history of humanity. Numerous trials (see [1-7]) have been done in order to evolve existing numeral systems and to include infinite and infinitesimal numbers in them. To emphasize importance of the subject it is sufficient to mention that the Continuum Hypothesis related to infinity has been included by David Hilbert as the problem number one in his famous list of 23 unsolved mathematical problems that have influenced strongly development of Mathematics and Computer Science in the XXth century (see [5]).

The point of view on infinity accepted nowadays is based on the famous ideas of Georg Cantor (see [1]) who has shown that there exist infinite sets having a different number of elements. However, it is well known that Cantor's approach leads to some paradoxes. The most famous and simple of them is, probably, Hilbert's paradox of the Grand Hotel (see, for example, [8]). Problems arise also in connection with the fact that usual arithmetical operations have been introduced for a finite number of operands.

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There exist different ways to generalize traditional arithmetic for finite numbers to the case of infinite and infinitesimal numbers (see [1,3,6,7,9]). However, arithmetics developed for infinite numbers are quite different with respect to the finite arithmetic we are used to deal with. Moreover, very often they leave many undetermined operations where infinite numbers take part (for example, infinity minus infinity, infinity divided by infinity, sum of infinitely many items, etc.) or use representation of infinite numbers based on infinite sequences of finite numbers. These crucial difficulties did not allow people to construct computers that would be able to work with infinite and infinitesimal numbers in the same manner as we are used to do with finite numbers.

In fact, in modern computers, only arithmetical operations with finite numbers are realized. Numbers can be represented in computer systems in various ways using positional numeral systems with a finite radix *b*. We remind that *numeral* is a symbol or group of symbols that represents a *number*. The difference between numerals and numbers is the same as the difference between the words and the things they refer to. A *number* is a concept that a *numeral* expresses. The same number can be represented by different numerals. For example, the symbols ‘3’, ‘three’, and ‘III’ are different numerals, but they all represent the same number.

Usually, when mathematicians deal with infinite objects (sets or processes) it is supposed that human beings are able to perform certain operations infinitely many times. For example, in a fixed numeral system it is possible to write down a numeral with *any* number of digits. However, this supposition is an abstraction (courageously declared by constructivists e.g, in [10]) because we live in a finite world and all human beings and/or computers finish operations they have started.

The point of view proposed in this paper does not use this abstraction and, therefore, is closer to the world of practical calculus than the traditional approaches. On one hand, we assume existence of infinite sets and processes. On the other hand, we accept that any of the existing numeral systems allows one to write down only a finite number of numerals and to do a finite number of operations. Thus, the problem we deal with can be formulated as follows: How to describe infinite sets and infinite processes by a finite number of symbols and how to do calculations with them?

The second important point in the paper is linked to the latter part of this question. The goal of the paper is to construct a new numeral system that would allow us to introduce and to treat infinite and infinitesimal numbers in the same manner as we are used to do with finite ones, i.e., by applying the philosophical principle of Ancient Greeks ‘*the part is less than the whole*’ which, in our opinion, reflects very well the world around us but is not incorporated in traditional infinity theories.

Of course, due to this more restrictive and applied statement, such concepts as bijection, numerable and continuum sets, cardinal and ordinal numbers cannot be used in this paper. However, the approach proposed here does not contradict Cantor. In contrast, it evolves his deep ideas regarding existence of different infinite numbers.

Let us start our consideration by studying situations arising in practice when it is necessary to operate with extremely large quantities (see [8] for a detailed discussion). Imagine that we are in a granary and the owner asks us to count how much grain he has inside it. There are a few possibilities of finding an answer to this question. The first one is to count the grain seed by seed. Of course, nobody can do this because the number of seeds is enormous.

To overcome this difficulty, people take sacks, fill them in with seeds, and count the number of sacks. It is important that nobody counts the number of seeds in a sack. At the end of the counting procedure, we shall have a number of sacks completely filled and some remaining seeds that are not sufficient to complete the next sack. At this moment it is possible to return to the seeds and to count the number of remaining seeds that have not been put in sacks (or a number of seeds that it is necessary to add to obtain the last completely full sack).

If the granary is huge and it becomes difficult to count the sacks, then trucks or even big train waggons are used. Of course, we suppose that all sacks contain the same number of seeds, all trucks – the same number of sacks, and all waggons – the same number of trucks. At the end of counting we obtain a result in the following form: the granary contains 16 waggons, 19 trucks, 12 sacks, and 4 seeds of grain. Note, that if we add, for example, one seed to the granary, we can count it and see that the granary has more grain. If we take out one waggon, we again be able to say how much grain has been subtracted.

Thus, in our example it is necessary to count large quantities. They are finite but it is impossible to count them directly using elementary units of measure, u_0, i.e., seeds, because the quantities expressed in these units would be too large. Therefore, people are forced to behave as if the quantities were infinite.

To solve the problem of 'infinite' quantities, new units of measure, u_1, u_2 and u_3 are introduced (units u_1 – sacks, u_2 – trucks, and u_3 – waggons). The new units have the following important peculiarity: it is not known how many units u_i there are in the unit u_{i+1} (we do not count how many seeds are in a sack, we just *complete* the sack). Every unit u_{i+1} is filled in completely by the units u_i. Thus, we know that all the units u_{i+1} contain a certain number K_i of units u_i but this number, K_i, is unknown. Naturally, it is supposed that K_i is the same for all instances of the units. Thus, numbers that it was impossible to express using only initial units of measure are perfectly expressible if new units are introduced.

This key idea of counting by introduction of new units of measure will be used in the paper to deal with infinite quantities. In Section 2, we introduce a new positional system with infinite radix allowing us to write down not only finite but infinite and infinitesimal numbers too. In Section 3, we describe arithmetical operations for all of them. Applications dealing with infinite sets, divergent series, and limits viewed from the positions of the new approach can be found in [8] and are not discussed in this paper. After all, Section 4 concludes the paper.

We conclude this Introduction by emphasizing that the goal of the paper is not to construct a complete theory of infinity and to discuss such concepts as, for example, 'set of all sets'. In contrast, the problem of infinity is considered from positions of applied mathematics and theory and practice of computations – fields being among the main scientific interests (see, for example, [8,11]) of the author. A new viewpoint on infinity and the corresponding mathematical and computer science tools are introduced in the paper in order to give possibilities to solve applied problems.

2. Infinite and infinitesimal numbers

Different numeral systems have been developed by humanity to describe finite numbers. More powerful numeral systems allow one to write down more numerals and, therefore, to express more numbers. However, in all existing numeral systems allowing us to do calculations numerals corresponding only to finite numbers are used. Thus, in order to have a possibility to write down infinite and infinitesimal numbers by a finite number of symbols, we need at least one new numeral expressing an infinite (or an infinitesimal) number. Then, it is necessary to propose a new numeral system fixing rules for writing down infinite and infinitesimal numerals and to describe arithmetical operations with them.

Note that introduction of a new numeral for expressing infinite and infinitesimal numbers is similar to introduction of the concept of zero and the numeral '0' that in the past have allowed people to develop positional systems being more powerful than numeral systems existing before.

In positional numeral systems fractional numbers are expressed by the record

$$\left(a_n a_{n-1} \dots a_1 a_0 . a_{-1} a_{-2} \dots a_{-(q-1)} a_{-q}\right)_b \tag{1}$$

where numerals a_i, $-q \le i \le n$ are called *digits*, belong to the alphabet $\{0,1,\dots,b–1\}$, and the dot is used to separate the fractional part from the integer one. Thus, the numeral (1) is equal to the sum

$$a_n b^n + a_{n-1} b^{n-1} + \dots + a_1 b^1 + a_0 b^0 + a_{-1} b^{-1} + \dots + a_{-(q-1)} b^{-(q-1)} + a_{-q} b^{-q} \tag{2}$$

In modern computers, the radix $b = 2$ with the alphabet $\{0,1\}$ is mainly used to represent numbers.

Record (1) uses numerals consisting of one symbol each, i.e., digits $a_i \in \{0,1,\dots,b–1\}$, to express how many finite units of the type b^i belong to the number (2). Quantities of finite units b^i are counted separately for each exponent i and all symbols in the alphabet $\{0,1,\ldots,b-1\}$ express finite numbers.

A new positional numeral system with infinite radix described in this section evolves the idea of separate count of units with different exponents used in traditional positional systems to the case of infinite and infinitesimal numbers. The infinite radix of the new system is introduced as the number of elements of the set N of natural numbers expressed by the numeral ① called *grossone*. This mathematical object is introduced by describing its properties postulated by the *Infinite Unit Axiom* consisting of three parts: Infinity, Identity, and Divisibility (we introduce them soon). This axiom is added to those for real numbers similarly to addition of the axiom determining zero to the axioms of natural numbers when integer numbers are introduced. This means that it is postulated that associative and commutative properties of multiplication and addition, distributive property of multiplication over addition, existence of inverse elements with respect to addition and multiplication hold for grossone as for finite numbers.

Note that usage of a numeral indicating totality of the elements we deal with is not new in mathematics. It is sufficient to remind the theory of probability where events can be defined in two ways. First, as union of elementary events; second, as a sample space, Ω, of all possible elementary events from where some elementary events have been excluded. Naturally, the second way to define events becomes particularly useful when the sample space consists of infinitely many elementary events.

– *Infinity.* For any finite natural number n it follows $n <$ ①.
– *Identity.* The following relations link ① to identity elements 0 and 1

$$0 \cdot ① = ① \cdot 0 = 0, \quad ① - ① = 0, \quad \frac{①}{①} = 1, \quad ①^0 = 1, \quad 1^{①} = 1. \tag{3}$$

– *Divisibility.* For any finite natural number n sets $\mathbb{N}_{k,n}$, $1 \le k \le n$ being the nth parts of the set, N, of natural numbers have the same number of elements indicated by the numeral ①$/n$ where

$$\mathbb{N}_{k,n} = \{k, k+n, k+2n, k+3n, \ldots\}, \quad 1 \le k \le n, \quad \bigcup_{k=1}^{n} \mathbb{N}_{k,n} = \mathbb{N}. \tag{4}$$

For example for $n = 1,2,3$ we have

$$① \rightarrow \mathbb{N} = \{1,2,3,4,5,6,7,\ldots\}$$

$$\frac{①}{2} \begin{array}{l} \nearrow \mathbb{N}_{1,2} = \{1, \quad 3, \quad 5, \quad 7, \quad \ldots\} \\ \searrow \mathbb{N}_{2,2} = \{\quad 2, \quad 4, \quad 6, \quad \ldots\} \end{array}$$

$$
\begin{array}{lllllll}
 & \mathbb{N}_{1,3} = & \{1, & & 4, & & 7, \ \dots \ \} \\
 \nearrow & & & & & & \\
\frac{①}{3} \rightarrow & \mathbb{N}_{2,3} = & \{ & 2, & & 5, & \dots \ \} \\
 \searrow & & & & & & \\
 & \mathbb{N}_{3,3} = & \{ & \quad 3, & & 6, & \dots \ \}
\end{array}
$$

Before the introduction of the new positional system let us see some properties of grossone. Its role in infinite arithmetic is similar to the role of number 1 in the finite one and it will serve us as the basis for construction of other infinite numbers. It is important to emphasize that to introduce ①/n we do not try to count elements k, $k+n$, $k+2n$, $k+3n$, … In fact, we cannot do this since our possibilities to count are limited and, therefore, we are not able to count for infinity. In contrast, we postulate following the above mentioned Ancient Greeks' principle *the part is less than the whole* (see [8,9,12,] for detailed discussions on such a kind of approaches) that the number of elements of the n th part of the set, i.e., ①/n, is n times less than the number of elements of the whole set, i.e., than ①. In terms of our granary example ① can be interpreted as the number of seeds in the sack. Then, if the sack contains ① seeds, its nth part contains ①/n seeds.

The numbers ①/n have been introduced as numbers of elements of sets $N_{k,n}$ thus, they are integer. For example, due to the introduced axiom, the set

$$\mathbb{N}_{2,5} = \{2, 7, 12, \dots\}$$

has ①/5 elements and the set

$$\mathbb{N}_{3,10} = \{3, 13, 23, \dots\}$$

has ①/10 elements.

The number of elements of sets being union, intersection, difference, or product of other sets of the type $N_{k,n}$ is defined in the same way as these operations are defined for finite sets. Thus, we can define the number of elements of sets being results of these operations with finite sets and infinite sets of the type $N_{k,n}$. For instance, the number of elements of the set

$$\mathbb{N}_{2,5} \cup \mathbb{N}_{3,10} \cup \{2, 3, 4, 5\}$$

is $\frac{①}{5} + \frac{①}{10} + 2$ because

$$\mathbb{N}_{2,5} \cap \mathbb{N}_{3,10} = \varnothing, \quad 2 \in \mathbb{N}_{2,5}, \quad 3 \in \mathbb{N}_{3,10}.$$

It is worthwhile noticing that, as it is for finite sets, infinite sets constructed using finite sets and infinite sets of the type $N_{k,n}$ have the same number of elements independently of objects outside the sets. A general rule for determining the number of elements of infinite sets having a more complex structure can be also given but it is not discussed in this paper (see [8]).

Introduction of the numeral ① allows us to write down the set, *N*, of natural numbers in the form

$$\mathbb{N} = \{1, 2, 3, \quad \ldots \quad ①-2,\ ①-1,\ ①\}. \tag{5}$$

It is worthwhile noticing that set (5) is the same set of natural numbers we are used to dealing. We have introduced grossone as the quantity of natural numbers. Thus, it is the biggest natural number and numbers

$$\ldots\ ①-3,\ ①-2,\ ①-1 \tag{6}$$

less than grossone are natural numbers as the numbers 1,2,3,… The difficulty to accept existence of infinite natural numbers is in the fact that traditional numeral systems did not allow us to see these numbers. Similarly, primitive tribes working with unary numeral system were able to see only numbers 1, 2, and 3 because they operated only with numerals *I*, *II*, *III* and did not suspect existence of other natural numbers. For them, all quantities bigger than *III* were just 'many' and such operations as $II + III$ and $I + III$ give the same result, i.e., 'many'. Note that this happens not because $II + III = I + III$ but due to weakness of this primitive numeral system. This weakness leads also to such results as 'many'+ 1 = 'many' and 'many'+ 2 = 'many' which are very familiar to us in the context of views on infinity used in the traditional calculus: $\infty + 1 = \infty,\ \infty + 2 = \infty$.

As an example let us consider a numeral system *S* able to express only numbers 1 and 2 by the numerals '1' and '2' (this system is even simpler than that of primitive tribes which was able to express three natural numbers). If we add to this system the new numeral ① it becomes possible to express the following numbers

$$\underbrace{1, 2}_{finite}, \quad \ldots \quad \underbrace{\frac{①}{2}-2, \frac{①}{2}-1, \frac{①}{2}, \frac{①}{2}+1, \frac{①}{2}+2}_{infinite}, \quad \ldots \quad \underbrace{①-2, ①-1, ①}_{infinite}.$$

In this record the first two numbers are finite, the remaining eight are infinite, and dots show the natural numbers that are not expressible in this numeral system. This numeral system does not allow us to do such operations as 2 + 2 or 2 + ①/2 + 2 because their results cannot be expressed in this system but, of course, we do not write that results of these operations are equal, we just say that the results are not expressible in this numeral system and it is necessary to take another, more powerful one.

The introduction of grossone allows us to obtain the following interesting result: the set *N* is not a monoid under addition. In fact, the operation ① + 1 gives us as the result a number greater than ①. Thus, by definition of grossone, ① + 1 does not belong to *N* and, therefore, *N* is not closed under addition and is not a monoid.

This result also means that adding the Infinite Unit Axiom to the axioms of natural numbers defines the set of *extended natural numbers* indicated as $\widehat{\mathbb{N}}$ and including $\mathbb{N}$ as a proper subset

$$\widehat{\mathbb{N}} = \{1, 2, \ldots, ①-1, ①, ①+1, \ldots, ①^2-1, ①^2, ①^2+1, \ldots\}.$$

Again, extended natural numbers greater than grossone can also be interpreted in the terms of sets of numbers. For example, ① + 3 as the number of elements of the set $\mathbb{N} \cup \{a,b,c\}$ where numbers $a,b,c \notin \mathbb{N}$ and $①^2$ as the number of elements of the set $\mathbb{N} \times \mathbb{N}$. In terms of our granary example ① + 3 can be interpreted as a sack plus three seeds and $①^2$ as a truck.

We have already started to write down simple infinite numbers and to do arithmetical operations with them without focusing our attention upon this question. Let us consider it systematically.

To express infinite and infinitesimal numbers we shall use records that are similar to (1) and (2) but have some peculiarities. In order to construct a number C in the new numeral positional system with base ① we subdivide C into groups corresponding to powers of ①:

$$C = c_{p_m} ①^{p_m} + \ldots + c_{p_1} ①^{p_1} + c_{p_0} ①^{p_0} + c_{p_{-1}} ①^{p_{-1}} + \ldots + c_{p_{-k}} ①^{p_{-k}}. \tag{7}$$

Then, the record

$$C = c_{p_m} ①^{p_m} \ldots c_{p_1} ①^{p_1} c_{p_0} ①^{p_0} c_{p_{-1}} ①^{p_{-1}} \ldots c_{p_{-k}} ①^{p_{-k}} \tag{8}$$

represents the number C, symbols c_i are called *grossdigits*, symbols p_i are called *grosspowers*. The numbers p_i are such that $p_i > 0$, $p_0 = 0$, $p_{-i} < 0$ and

$$p_m > p_{m-1} > \ldots p_2 > p_1 > p_{-1} > p_{-2} > \ldots p_{-(k-1)} > p_{-k}$$

In the traditional record (1) there exists a convention that a digit a_i shows how many powers b_i are present in the number and the radix b is not written explicitly. In record (8) we write $①^{p_i}$ explicitly because in the new numeral positional system the number i in general is not equal to the grosspower p_i. This gives possibility to write, for example, such numbers as $7①^{244.5}\, 3①^{-32}$ where $p_1 = 244.5$, $p_{-1} = -32$.

Finite numbers in this new numeral system are represented by numerals having only one grosspower equal to zero. In fact, if we have a number C such that $m = k = 0$ in representation (8), then due to (3) we have $C = c_0①^0\, c_0$. Thus, the number C in this case does not contain infinite and infinitesimal units and is equal to the grossdigit c_0 which being a conventional finite number can be expressed in the form (1), (2) by any positional system with finite base b (or by another numeral system). It is important to emphasize that the grossdigit c_0 can be integer or fractional and can be expressed by a few symbols in contrast to the traditional record (1) where each digit is integer and is represented by one symbol from the alphabet $\{0,1,2,\ldots,b-1\}$. Thus, the grossdigit c_0 shows how many finite units and/or parts of the finite unit, $1 = ①^0$, there are in the number C. Grossdigits can be written in positional systems, in the form p/q where p and q are integer numbers, or in any other finite numeral system.

Analogously, in the general case, all grossdigits $c_i\, -k \le i \le m$ can be integer or fractional and expressed by many symbols. For example, the number $(7/3)①^4(84/19)①^{-3.1}$ has grossdigits $c_4 = 7/3$ and $c_{-3.1} = 84/19$. All grossdigits can also be negative; they show how many corresponding units should be added or subtracted in order to form the number C.

Infinite numbers are written in this numeral system as numerals having grosspowers greater than zero, for example $7①^{244.5}3①^{-32}$ and $-2①^{74}3①^{0}37①^{-2}11①^{-15}$ are infinite numbers. In the following example the left-hand expression presents the way of writing down infinite numbers and the right-hand shows how the value of the number is calculated:

$$15①^{14}17.2045①^{3}\left(-52.1\right)①^{-6} = 15①^{14} + 17.2045①^{3} - 52.1①^{-6}$$

If a grossdigit c_{p_i} is equal to 1 then we write $①^{p_i}$ instead of $1①^{p_i}$. Analogously, if power $①^0$ is the lowest in a number then we often use simply the corresponding grossdigit c_0 without $①^0$, for instance, we write $23①^{14}5$ instead of $23①^{14}5①^0$ or 3 instead of $3①^0$.

Numerals having only negative grosspowers represent infinitesimal numbers. The simplest number from this group is $①^{-1} = 1/①$ being the inverse element with respect to multiplication for ①:

$$\frac{1}{①} \cdot ① = 1 \cdot \frac{1}{①} = 1 \tag{9}$$

Note that all infinitesimals are not equal to zero. Particularly, $1/① > 0$ because $1 > 0$ and $① > 0$. It has a clear interpretation in our granary example. Namely, if we have a sack and it contains ① seeds then one sack divided by ① is equal to one seed. Vice versa, one seed, i.e., $1/①$, multiplied by the number of seeds in the sack, ①, gives one sack of seeds.

Inverse elements of more complex numbers including grosspowers of ① are defined by a complete analogy. The following two numbers are examples of infinitesimals $3①^{-32}$, $37①^{-2}$, $(-11)①^{-15}$.

3. Arithmetical operations with infinite, infinitesimal, and finite numbers

Let us now introduce arithmetical operations for infinite, infinitesimal, and finite numbers. The operation of *addition* of two given infinite numbers A and B (the operation of subtraction is a direct consequence of that of addition and is thus omitted) returns as the result an infinite number C

$$A = \sum_{i=1}^{K} a_{k_i} ①^{k_i}, \quad B = \sum_{j=1}^{M} b_{m_j} ①^{m_j}, \quad C = \sum_{i=1}^{L} c_{l_i} ①^{l_i}, \tag{10}$$

where C is constructed by including in it all items $a_{k_i}①^{k_i}$ from A such that $k_i \neq m_j$, $1 \leq j \leq M$ and all items $b_{m_j}①^{m_j}$ from B such that $m_j \neq k_i$, $1 \leq i \leq K$. If in A and B there are items such that $k_i = m_j$ for some i and j then this grosspower k_i is included in C with the grossdigit $b_{k_i} + a_{k_i}$, i.e., as $\left(b_{k_i} + a_{k_i}\right)①^{k_i}$. It can be seen from this definition that the introduced operation enjoys the usual properties of commutativity and associativity due to definition of grossdigits and the fact that addition for each grosspower of ① is done separately.

Let us illustrate the rules by an example (in order to simplify the presentation in all the following examples the radix $b = 10$ is used for writing down grossdigits). We consider two infinite numbers A and B where

$$A = 16.5①^{44.2}(-12)①^{12}17①^{0}1.17①^{-3}$$

$$B = 23①^{14}6.23①^{3}10.1①^{0}(-1.17)①^{-3}11①^{-43}$$

Their sum C is calculated as follows

$$\begin{aligned} C = A + B = &16.5①^{44.2} + (-12)①^{12} + 17①^{0} + 1.17①^{-3} + \\ &23①^{14} + 6.23①^{3} + 10.1①^{0} - 1.17①^{-3} + 11①^{-43} = \\ &16.5①^{44.2} + 23①^{14} - 12①^{12} + 6.23①^{3} + \\ &(17 + 10.1)①^{0} + (1.17 - 1.17)①^{-3} + 11①^{-43} = \\ &16.5①^{44.2} + 23①^{14} - 12①^{12} + 6.23①^{3} + 27.1①^{0} + 11①^{-43} = \\ &16.5①^{44.2}23①^{14}(-12)①^{12}6.23①^{3}27.1①^{0}11①^{-43} \end{aligned}$$

The operation of *multiplication* of two given infinite numbers A and B from (10) returns and as a result the infinite number C is constructed as follows.

$$C = \sum_{j=1}^{M} C_j, \quad C_j = b_{m_j}①^{m_j} \cdot A = \sum_{i=1}^{K} a_{k_i} b_{m_j} ①^{k_i + m_j}, \quad 1 \leq j \leq M \tag{11}$$

Similarly to addition, the introduced multiplication is commutative and associative. It is easy to show that the distributive property is also valid for these operations.

Let us illustrate this operation by the following example. We consider two infinite numbers

$$A = ①^{18}(-5)①^{2}(-3)①^{1}0.2, \quad B = ①^{2}(-1)①^{1}7①^{-3}$$

and calculate the product $C = B \cdot A$. The first partial product C_1 is equal to

$$\begin{aligned} C_1 = 7①^{-3} \cdot A = 7①^{-3}\left(①^{18} - 5①^{2} - 3①^{1} + 0.2\right) = \\ 7①^{15} - 35①^{-1} - 21①^{-2} + 1.4①^{-3} = 7①^{15}(-35)①^{-1}(-21)①^{-2}1.4①^{-3}. \end{aligned}$$

The other two partial products, C_2 and C_3, are computed analogously:

$$C_2 = -①^1 \cdot A = -①^1\left(①^{18} - 5①^2 - 3①^1 + 0.2\right) = -①^{19} 5①^3 3①^2 (-0.2)①^1,$$

$$C_3 = ①^2 \cdot A = ①^2\left(①^{18} - 5①^2 - 3①^1 + 0.2\right) = ①^{20}(-5)①^4(-3)①^3 0.2①^2.$$

Finally, by taking into account that grosspowers $①^3$ and $①^2$ belong to both C_2 and C_3 and, therefore, it is necessary to sum up the corresponding grossdigits, the product C is equal (due to its length, the number C is written in two lines) to

$$C = C_1 + C_2 + C_3 = ①^{20}(-1)①^{19} 7①^{15}(-5)①^4 2①^3 3.2①^2$$
$$(-0.2)①^1(-35)①^{-1}(-21)①^{-2} 1.4①^{-3}.$$

In the operation of *division* of a given infinite number C by an infinite number B we obtain an infinite number A and a reminder R that can be also equal to zero, i.e., $C = A \cdot B + R$.

The number A is constructed as follows. The numbers B and C are represented in the form (10). The first grossdigit a_{k_K} and the corresponding maximal exponent k_K are established from the equalities

$$a_{k_K} = c_{l_L} / b_{m_M}, \quad k_K = l_L - m_M. \tag{12}$$

Then the first partial reminder R_1 is calculated as

$$R_1 = C - a_{k_K} ①^{k_K} \cdot B. \tag{13}$$

If $R_1 \neq 0$ then the number C is substituted by R_1 and the process is repeated by a complete analogy. The grossdigit $a_{k_{K-i}}$, the corresponding grosspower k_{K-1} and the partial reminder R_{i+1} are computed by formulae (14) and (15) obtained from (12) and (13) as follows: l_L and c_{l_L} are substituted by the highest grosspower n_i and the corresponding grossdigit r_{n_i} of the partial reminder R_i that in its turn substitutes C:

$$a_{k_{K-i}} = r_{n_i} / b_{m_M}, \quad k_{K-i} = n_i - m_M \tag{14}$$

$$R_{i+1} = R_i - a_{k_{K-i}} 172^{k_{K-i}} \cdot B, \quad i \geq 1 \tag{15}$$

The process stops when a partial reminder equal to zero is found (this means that the final reminder $R = 0$) or when a required accuracy of the result is reached.

The operation of division will be illustrated by two examples. In the first example we divide the number $C = -10①^3 16①^0 42①^{-3}$ by the number $B = 5①^3 7$. For these numbers we have

$$l_L = 3, \quad m_M = 3, \quad c_{l_L} = -10, \quad b_{m_M} = 5$$

It follows immediately from (12) that $a_{k_K} ①^{k_K} = -2①^0$. The first partial reminder R_1 is calculated as

$$R_1 = -10①^3 16①^0 42①^{-3} - \left(-2①^0\right) \cdot 5①^3 7 =$$
$$-10①^3 16①^0 42①^{-3} + 10①^3 14①^0 = 30①^0 42①^{-3}.$$

By a complete analogy we should construct $a_{k_{K-1}} ①^{k_{K-1}}$ by rewriting (12) for R_1. By doing so we obtain equalities

$$30 = a_{k_{K-1}} \cdot 5, \quad 0 = k_{K-1} + 3$$

and, as the result, $a_{k_{K-1}} ①^{k_{K-1}} = 6①^{-3}$. The second partial reminder is

$$R_2 = R_1 - 6①^{-3} \cdot 5①^3 7 = 30①^0 42①^{-3} - 30①^0 42①^{-3} = 0.$$

Thus, we can conclude that the reminder $R = R_2 = 0$ and the final result of division is $A = -2①^0 6①^{-3}$.

Let us now substitute the grossdigit 42 by 40 in C and divide this new number $\widetilde{C} = -10①^3 16①^0 40①^{-3}$ by the same number $B = 5①^3 7$. This operation gives us the same result $\widetilde{A}_2 = A = -2①^0 6①^{-3}$ (where subscript 2 indicates that two partial reminders have been obtained) but with the reminder $\widetilde{R} = \widetilde{R}_2 = -2①^{-3}$. Thus, we obtain $\widetilde{C} = B \cdot \widetilde{A}_2 + \widetilde{R}_2$. If we want to continue the procedure of division, we obtain $\widetilde{A}_3 = -2①^0 6①^{-3} (-0.4)①^{-6}$ with the reminder $\widetilde{R}_3 = 0.28①^{-6}$. Naturally, it follows $\widetilde{C} = B \cdot \widetilde{A}_3 + \widetilde{R}_3$. The process continues until a partial reminder $\widetilde{R}_i = 0$ is found or when a required accuracy of the result will be reached.

In all the examples above we have used grosspowers being finite numbers. However, all the arithmetical operations work by a complete analogy also for grosspowers being themselves numbers of the type (8). For example, if

$$X = 16.5①^{44.2①1.17①^{-3}} (-12)①^{12①} 1.17①^{-3},$$
$$Y = 23①^{44.2①1.17①^{-3}} (-1.17)①^{-3} 11①^{4①^{-23}},$$

then their sum Z is calculated as follows

$$Z = X + Y = 39.5①^{44.2①1.17①^{-3}} (-12)①^{12172} 11①^{4①^{-23}}$$

4. Conclusions

In this paper, a new positional numeral system with infinite radix has been described. This system allows us to express by a finite number of symbols not only finite numbers but infinite and infinitesimals too. All of them can be viewed as particular cases of a general framework used to express numbers.

It has been shown in [13] that the new approach allows us to construct a new type of computer – Infinity Computer – able to operate with finite, infinite, and infinitesimals quantities. Numerous examples dealing with infinite sets and processes, divergent series, limits, and measure theory viewed from the positions of the new approach can be found in [8]. We conclude this paper just by one example showing the potential of the new approach. We consider two infinite series $S_1 = 1+1+1+...$ and $S_2 = 3+3+3+...$ The traditional analysis gives us a very poor answer that both of them diverge to infinity. Such operations as $S_1 - S_2$ or S_1/S_2 are not defined.

In our terminology divergent series do not exist. Now, when we are able to express not only different finite numbers but also different infinite numbers, the records S_1 and S_2 are not well defined. It is necessary to indicate explicitly the number of items in the sum and it is not important if it finite or infinite. To calculate the sum it is necessary that the number of items and the result are expressible in the numeral system used for calculations. It is important to notice that even though a sequence cannot have more than ① elements the number of items in a series can be greater than grossone because the process of summing up is not necessary. This should be done by a sequential adding items.

Suppose that the series S_1 has k items and S_2 has n items:

$$S_1(k) = \underbrace{1+1+1+...+1}_{k}, \quad S_2(n) = \underbrace{3+3+3+...+3}_{n}$$

Then $S_1(k) = k$ and $S_2(n) = 3n$ for any values of k and n – finite or infinite. By giving numerical values to k and n we obtain numerical values for the sums. If, for instance, $k = n =$ ① then we obtain $S_1(5①) = 5①$, $S_2(5①) = 15①$ and

$$S_2(5①) - S_1(5①) = 10① > 0 .$$

If $k = 5①$ and $n =$ ① we obtain $S_1(5①) = 5①$, $S_2(①) = 3①$ and it follows

$$S_2(①) - S_1(5①) = -2① < 0 .$$

If $k = 3①$ and $n =$ ① we obtain $S_1(3①) = 3①$, $S_2(①) = 3①$ and it follows

$$S_2(①) - S_1(3①) = 0$$

Analogously, the expression $S_1(k)/S_2(n)$ can be calculated.

References

[1] Cantor G., *Contributions to the founding of the theory of transfinite numbers*. Dover Publications, New York, (1955).
[2] Cohen P.J., *Set theory and the Continuum Hypothesis*, Benjamin. New York, (1966).
[3] Conway J.H., Guy R.K., *The book of numbers*. Springer-Verlag, New York, (1996).
[4] Gödel K., *The consistency of the Continuum-Hypothesis*. Princeton University Press, Princeton, (1940).
[5] Hilbert D., *Mathematical problems: Lecture delivered before the International Congress of Mathematicians at Paris in 1900*. Bulletin of the American Mathematical Society, 8 (1902) 437.
[6] Loeb P.A., Wolff M.P.H., *Nonstandard analysis for the working mathematician*. Kluwer Academic Publishers, Dordrecht, (2000).
[7] Robinson A., *Non-standard analysis*. Princeton University Press, Princeton, (1996).
[8] Sergeyev Ya.D., *Arithmetic of infinity*. Edizioni Orizzonti Meridionali, CS, (2003).

[9] Benci V., Di Nasso M., *Numerosities of labeled sets: a new way of counting*. Advances in Mathematics, 173 (2003) 50.

[10] Markov A.A. Jr., Nagorny N.M., *Theory of algorithms*, second ed. . FAZIS, Moscow, (1996).

[11] Strongin R.G., Sergeyev Ya.D., *Global optimization and non-convex constraints: Sequential and parallel algorithms*. Kluwer Academic Publishers, Dordrecht, (2000).

[12] Mayberry J.P., *The foundations of mathematics in the theory of sets*. Cambridge University Press, Cambridge, (2001).

[13] Sergeyev Ya.D., *Computer system for storing infinite, infinitesimal, and finite quantities and executing arithmetical operations with them*. patent application, 08.03.04.