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Two hierarchies of *R*-recursive functions

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Abstract

In the paper some aspects of complexity of *R*-recursive functions are considered. The limit hierarchy of *R*-recursive functions is introduced by the analogy to the μ-hierarchy. Then its properties and relations to the μ-hierarchy are analysed.

1. Introduction

The classical theory of computation deals with the functions on enumerable (especially natural) domains. The fundamental notion in this field is the notion of a (partial) recursive function. The problem of hierarchies for these functions is also in the interest of mathematicians (for elementary, primitive recursive function, Grzegorczyk hierarchy, compare [1].

During past years many mathematicians have been interested in creating analogous models of computation on real numbers (see for example Grzegorczyk [2], Blum, Shub, Smale [3]). An interesting approach was given by Moore. In the work [4] he defined a set of functions on the reals *R* (called *R*-recursive functions) in the analogous way to the classical recursive functions on the natural numbers *N*. His model has a continuous time of computation (a continuous integration instead of a discrete recursion). The great importance in Moore's model has the zero-finding operation μ, which is used to construct μ-hierarchy of *R*-recursive functions.

It was shown [5] that the zero-finding operator μ can be replaced by the operation of infinite limits. This allows us to define a limit hierarchy and relate it to μ-hierarchy.

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2. Preliminaries

We start with a fundamental definition of a class of real functions called ***R***-recursive functions [4].

Definition 2.1 *The set of **R**-recursive functions is generated from the constants 0,1 by the operations:*

1) *composition:* $h(\overline{x}) = f(g(\overline{x}))$;
2) *differential recursion:* $h(\overline{x},0) = f(\overline{x}), \partial_y h(\overline{x},y) = g(\overline{x},y,h(\overline{x},y))$ *(the equivalent formulation can be given by integrals:* $h(\overline{x},y) = f(\overline{x}) + \int_0^y g(\overline{x},y',h(\overline{x},y'))dy'$*);*
3) m*-recursion* $h(\overline{x}) = m_y f(\overline{x},y) = \inf\{y : f(\overline{x},y) = 0\}$, *where infimum chooses the number **y** with the smallest absolute value and for two **y** with the same absolute value the negative one;*
4) *vector-valued functions can be defined by defining their components.*

Several comments are needed to the above definition. A solution of a differential equation need not be unique or can diverge. Hence, we assume that if h is defined by a differential recursion then h is defined only where a finite and unique solution exists. This is why the set of ***R***-recursive functions includes also partial functions. We use (after [4]) the name of ***R***-recursive functions in the article, however we should remember that in reality we have partiality here (partial ***R***-recursive functions).

The second problem arises with the operation of infimum. Let us observe that if an infinite number of zeros accumulates just above some positive *y* or just below some negative *y* then the infimum operation returns that *y* even if it itself is not a zero.

In the papers [5, 6] it was shown that if in the Moore's definition [4] m-operation is replaced by infinite limits: $h(\overline{x}) = \liminf_{y\to\infty} g(\overline{x},y)$, $h(\overline{x}) = \limsup_{y\to\infty} g(\overline{x},y)$ then the resulting class of functions remains the same.

This gives us also the following result (including the limit operation in the form $h(\overline{x}) = \lim_{y\to\infty} g(\overline{x},y)$, which can be in the obvious way obtained from limsup, liminf:

Corollary 2.2 *The class of **R**-recursive functions is closed under the operations of infinite limits:* $h(\bar{x}) = \liminf_{y\to\infty} g(\bar{x}, y)$, $h(\bar{x}) = \limsup_{y\to\infty} g(\bar{x}, y)$, $h(\bar{x}) = \lim_{y\to\infty} g(\bar{x}, y)$.

3. Hierarchies

The operator m is a key operator in generating the ***R***-recursive functions. In a physical sense it has a property of being strongly uncomputable. This fact suggests creating a hierarchy, which is built with respect to the number of uses of m in the definition of a given *f*.

Definition 3.1 *([4]) For a given **R**-recursive expression* $s(\bar{x})$, *let* $M_{x_i}(s)$ *(the* m*-number with respect to* x_i*) be defined as follows:*

$$M_x(0) = M_x(1) = M_x(-1) = 0, \tag{1}$$

$$M_x\left(f(g_1, g_2, \ldots)\right) = \max_j \left(M_{x_j}(f) + M_x(g_j)\right), \tag{2}$$

$$M_x\left(h = f + \int_0^y g(\bar{x}, y', h)\,dy'\right) = \max\left(M_x(f), M_x(g), M_h(g)\right), \tag{3}$$

$$M_y\left(h = f + \int_0^y g(\bar{x}, y', h)\,dy'\right) = \max\left(M_{y'}(g), M_h(g)\right), \tag{4}$$

$$M_x\left(m_y f(\bar{x}, y)\right) = \max\left(M_x(f), M_y(f)\right) + 1, \tag{5}$$

where x *can be any* $x_1, \ldots, x_n$ *for* $\bar{x} = (x_1, \ldots, x_n)$.

For an ***R***-recursive function *f*, let $M(f) = \max_{x_i}(s)$ minimized over all expressions ***s*** that define *f*. Now we are ready to define M-hierarchy (m-hierachy) as a family of $M_j = \{f : M'(f) \le j\}$.

Let us construct the analogous definition of L-hierarchy by replacing in the above definition M_x by L_x and changing line (5) to the following form (5'):

$$L_x\left(\liminf_{y\to\infty} g(\bar{x}, y)\right) = L_x\left(\limsup_{y\to\infty} g(\bar{x}, y)\right) =$$
$$= L_x\left(\lim_{y\to\infty} g(\bar{x}, y)\right) = \max\left(L_x(f), L_y(f)\right) + 1.$$

For an ***R***-recursive function *f*, let $L(f) = \max_i L_{x_i}(s)$ minimized over all expressions ***s*** that define ***f*** without using the m-operation.

Definition 3.2 *The **L**-hierarchy is a family of* $L_j = \{ f : L(f) \le j \}$.

Let us add that in Definition 3.2 we use explicitly the operator $f(\bar{x}) = \lim_{y\to\infty} g(\bar{x}, y)$ to avoid its construction by other operators (lim sup, lim inf), which would efect in a superficially higher class of a complexity of a function *f*.

As an obvious corollary from definitions we have the following statement.

Lemma 3.3 *The classes* M_0 *and* M_1 *are identical.*

A function $f \in L_0 = M_0$ will be called (by an analogy to the case of natural recursive functions) a primitive ***R***-recursive function. After Moore [4] we can conclude that such functions as: $-x$, $x+y$, xy, x/y, e^x, $\ln x$, y^x, $\sin x$, $\cos x$ are primitive ***R***-recursive.

We can give a few results on some levels of the limit hierarchy.

Lemma 3.4. *The Kronecker d function, the signum function and absolute value belong to the first level (* L_1 *)of limit hierarchy.*

Proof. It is sufficient to take the following definitions [5]: hence $d(0)=1$ and for all $x \ne 0$ we have $d(x)=0$ let us define $d(x) = \lim\inf_{y\to\infty}\left(\frac{1}{1+x^2}\right)^y$. Now from the expression $\lim\inf_{y\to\infty} \arctan xy = \begin{cases} p/2, & \text{if } x>0, \\ 0, & \text{if } x=0, \\ -p/2, & \text{if } x<0, \end{cases}$ we obtain $\operatorname{sgn}(x) = \frac{\lim\inf_{y\to\infty} \arctan xy}{2\arctan 1}$ and $|x| = \operatorname{sgn}(x)\,x$.

We should be careful with definions of functions by cases:

Lemma 3.5 *For* $h(\bar{x}) = \begin{cases} g_1(\bar{x}), & \textit{if } f(\bar{x}) = 0, \\ g_2(\bar{x}), & \textit{if } f(\bar{x}) = 1, \\ \vdots & \vdots \\ g_k(\bar{x}), & \textit{if } f(\bar{x}) \ge k-1 \end{cases}$ *and* $g_i \in L_{n_i}$ *for all* $1 \le i \le k$, $f \in L_m$ *the function* h *belongs to* $L_{\max(n_1,\dots n_k, m+1)}$

Proof. Let us see that $eq(x,y)=d(x-y)\in L_1$ and $ge(x,y)=\frac{(\operatorname{sgn}(x-y)+eq(x,y))}{2}+\frac{1}{2}\in L_1$. Then of course $h(\overline{x})=\sum_{i=1}^{k-1} g_i(\overline{x})eq(f(\overline{x}),i-1)+g_k(\overline{x})ge(f(\overline{x}),k-1)$.□

Of course this result can be easily extended to other forms of definitions by cases.

Lemma 3.6 *The function $\Theta(x)$ (equal to 1 if $x\geq 0$, otherwise 0), maximum $\max(x,y)$, square-wave function **s** are in L_2, the function $p(x)$ such that $p(x)=1$ for $x\in[2n,2n+1]$ and $p(x)=0$ for $x\in[2n+1,2n+2]$ is in L_2 and the floor function $\lfloor x\rfloor$ is in L_3.*

Proof. We give the proper definitions (from [6]) for these functions. Let

$$\Theta(x)=d(x-|x|),$$
$$\max(x,y)=xd(x-y)+(1-d(x-y))[x\Theta(x-y)+y\Theta(y-x)],$$
$$s(x)=\Theta(\sin(p\,x)).$$

The function $p(x)$ can be given as $s(x)\left(1-d\left(\sin\frac{(x-1)p}{2}\right)\right)$, so $p\in L_2$.

The floor function we can define by the auxiliary function $w(0)=0$, $\partial_x w(x)=2\Theta(-\sin(2p\,x))$ as

$$\lfloor x\rfloor=\begin{cases}2w(x/2) & \text{if } p(x)=1,\\ 2w((x-1)/2) & \text{if } p(x)=0.\end{cases}$$

From the above equation we have $\lfloor x\rfloor$ in L_3.□

Let us recall that if $f:R^n\rightarrow R$ is an ***R***-recursive function then the function $f_{iter}(i,\overline{x})$ is ***R***-recursive, too.

Lemma 3.7 *Let $f:R^n\rightarrow R$ belongs to the class L_i, then we have $f_{iter}:R^{n+1}\rightarrow R$ is in $L_{\max(2,j)}$.*

Proof. The definitions, which were given by Moore [3] $f_{iter}(i,\overline{x})=h(2i)$, where

$$h(0)=g(0)=\overline{x},$$

$$\partial_t g(t) = \left[f(h(t)) - h(t) \right] s(t),$$
$$\partial_t h(t) \Rightarrow \geq \left[\frac{g(t)-h(t)}{r(t)} \right] (1 - s(t)),$$

with s - a square wave function in L_2 and $r(0)=0$, $\partial_t r(t) = 2s(t) - 1$, $r, s \in L_2$ give us the desirable statement. □

Lemma 3.8 *The* $\boldsymbol{R}^l$*-recursive functions* $g_2 : R^2 \to R$, $g_2^1, g_2^2 : R \to R$ *such that* $(\forall x, y \in R) g_2^1(g_2(x,y)) = x$, $(\forall x, y \in R) g_2^2(g_2(x,y)) = y$, *have the following properties:* g_2, g_2^1 *are in* L_{10}, g_2^2 *is in* L_{14}.

Proof. We have the auxiliary functions Γ_2, Γ_2^1, Γ_2^2, which are coding and decoding functions in the interval $(0,1) : \Gamma_2(x,y) = c(x) + c(y)/10$, where

$$c(x) = \lim_{i\to\infty} z(a(i,x)) / 10^{2i} + b(i,x)/10^i ,$$

and later $z(x) = \lim_{i\to\infty} z'_{iter}(i,x)$,

$$z'_{iter}(i, a_1 \ldots a_n . a_{n+1} \ldots) = a_1 \ldots a_n 0 \ldots a_{n+1} 0 . a_{n+i+1} \ldots,$$
$$a(i, 0.a_1 a_2 \ldots a_i \ldots) = 0.a_1 \ldots a_i$$
$$b(i, 0.a_1 a_2 \ldots a_i \ldots) = 0.\underbrace{0}_{i} 0 a_{i+1} \ldots,$$

$(z'(x) = \begin{cases} 100\lfloor x \rfloor + 10(x - \lfloor x \rfloor), & \text{if } \lfloor x \rfloor \neq x, \\ x, & \text{if } \lfloor x \rfloor = x; \end{cases} \in L_4, a, b \in L_4$. Also z'_{iter} belongs to L_4, hence $\Gamma_2(x,y) \in L_{10}$, decoding of the first element is described in the symmetric way so $\Gamma_2^1(x)$ is in L_{10}, but $\Gamma_2^2(x) = \Gamma_2^1(10 - \lfloor 10x \rfloor)$ so $\Gamma_2^2 \in L_{14}$.

The functions Γ_2, Γ_2^1, Γ_2^2 can be extended to all reals by one-to-one $f:(0,1) \to R \in L_0$ without the loss of their class. □

The same method of coding and decoding by interlacing of ciphers (only the power of 10 should be changed) gives us the functions $g_n : R^n \to R$ and $g_n^i : R \to R$ for $i = 1, \ldots, n$ such that

$$(\forall i)(\forall x_1 \ldots, x_n \in R) g_n^i(g_n(x_1, \ldots, x_n)) = x_i$$

in the same class: $g_n, g_n^1 \in L_{10}$ and $(\forall i > 1) g_n^i \in L_{14}$.

We finish this part with the important form of defining: a new function is given as a product of values $\boldsymbol{f}$ in some integer points.

Lemma 3.9 *There exists such constant* $p \in N$ *that for the function*

$$\prod_{z=0}^{y} f(\bar{x},z) = \begin{cases} f(\bar{x},0)\, f(\bar{x},1) \ldots f(\bar{x},\lfloor y-1 \rfloor), & \text{if } y \geq 1, \\ 1, & \text{if } 0 \leq y < 1, \\ 0, & \text{if } y < 0, \end{cases}$$

if the function f is in the class L_m *then* $\prod_{z=0}^{y} f(\bar{x},z)$ *is in the class* L_{m+p} *(p is independent of m).*

Proof. By the definitions

$$t(w) = g_{n+2}\left(g_{n+2}^{1,n}(w), g_{n+2}^{n+1}(w)+1, f\left(g_{n+2}^{1,n}(w), g_{n+2}^{n+1}(w)\right)\cdot g_{n+2}^{n+2}(w)\right)$$

and

$$S(\bar{x},z) = \underbrace{t(\ldots t}_{\lfloor z \rfloor}(s(\bar{x},0))\ldots) = t_{iter}\left(\lfloor z \rfloor, g_{n+2}(\bar{x},0,1)\right)$$

we get the property

$$\prod_{y=0}^{z} f(\bar{x},y) = g_{n+2}^{n+2}\left(S(\bar{x},z)\right).$$

From the defintion of the limit hierarchy we get $\prod_{y=0}^{z} f(\bar{x},y) \in L_{m+38}$.□

In the rest of the paper we will use the constant *p* as the number of limits used in the recursive defintion of the product $\prod_{y=0}^{z} f(\bar{x},y)$ instead of the value 38. The above constructions are tedious and can be improved with a better approximation of *p*.

4. Main results

Now we are ready to formulate two theorems which demonstrate connections between L-hierarchy and M-hierarchy.

Theorem 4.1 *Let* $f : R^n \to R$ *be an **R**-recursive function. Then if* $f \in L_i$ *then* $f \in M_{10i}$.

Proof. We use a simple induction here. The case $i = 0$ is given in Lemma 3.3. Now let us suppose that the thesis is true for $i = n$. Let $f \in L_{n+1}$ be defined as $f(\bar{x}) = \lim_{y\to\infty} g(\bar{x},y)$ for $g \in L_n$. Then we can recall Theorem 4.2 from [6] which gives us the following result: to define ***f*** from *g* it is necessary to use at most 10 m-operation. Hence for $g \in M_{10n}$ the function $\boldsymbol{f}$ satisfies $f \in M_{10n+10}$. Similar inferences hold for lim inf, lim sup. □

Now we can give the result about the 'limit complexity' of the infimum operator m.

Lemma 4.2 *If* $f(\bar{x}, y): R^{n+1} \to R$ *is in the class* L_m *then the function* $g: R^n \to R$, $g(\bar{x}) = m_y f(\bar{x}, y)$ *is in the class* L_{m+3p+9} *is from Lemma 3.9.*

Proof. Here we must employ the results from [6]. There we defined the function $g: R^n \to R$, $g(\bar{x}) = m_y f(\bar{x}, y)$ for $f(\bar{x}, y): R^{n+1} \to R$ (f - $\boldsymbol{R}$-recursive) replacing the m-operator by limit operation. First we introduced the function

$$Z^f(\bar{x}, z) = \begin{cases} \inf_y \{ f : K^f(\bar{x}, y) = 0 \}, & \text{if } z = 0 \text{ and } \exists y K^f(\bar{x}, y) = 0, \\ \text{undefined} & \text{if } z = 0 \text{ and } \forall y K^f(\bar{x}, y) \neq 0, \\ 1 & \text{if } z \neq 0, \end{cases}$$

given in the following way:

$$Z^f(\bar{x}, z) = \begin{cases} \text{undefined} & \text{if } (z=0) \wedge \left(S^f(\bar{x}) < 1/12 \right), \\ \sqrt{S^f(\bar{x}) - 1/12}, & \text{if } (z=0) \wedge \left(S^f(\bar{x}) \geq 1/12 \right) \\ & \wedge f\left(\bar{x}, \sqrt{S^f(\bar{x}) - 1/12} \right) = 0, \\ -\sqrt{S^f(\bar{x}) - 1/12}, & \text{if } (z=0) \wedge \left(S^f(\bar{x}) \geq 1/12 \right) \\ & \wedge f\left(\bar{x}, -\sqrt{S^f(\bar{x}) - 1/12} \right) = 0, \\ 1, & \text{if } z \neq 0. \end{cases}$$

where $S^f(\bar{x}) = \lim_{t\to\infty} S_1^f(\bar{x}, t) + \lim_{t\to\infty} S_2^f(\bar{x}, t)$. Both functions S_1^f, S_2^f are defined by an integration

$$S_i^f(\bar{x}, t) = \int y^2 \left(1 - h^f\left(\bar{x}, (-1)^{i+1} y - 1/2, (-1)^{i+1} y + 1/2 \right) \right) dy, \ i = 1, 2$$

from $h^f(\bar{x}, a, b) = \liminf_{t\to\infty} \prod_{w=0}^{z+1} K^f\left(\bar{x}, a + w\frac{b-a}{z} \right)$ where K^f is the characteristic function of f.

Hence we can conclude that if K^f is in the L_s then Z_f is in the class L_{s+p+3}. Let us finish with the definition of the characteristic function of the infimum of zeros of f (see Theorem 4.2 from [5]

$$K_m^f(y) = 1 - \lim_{a\to-\infty}\lim_{b\to\infty}\lim_{z\to\infty} G^f(\overline{x}, z, a, b, y),$$

where $G^f(\overline{x}, z, a, b, y)$ divides the interval $[a,b]$ into $2^{\lfloor z\rfloor}$ equal subintervals and gives the value 1 for y from the subintervals, which contains the least zero of f in $[a,b]$ and value 0 otherwise. Precisely for y from $\left[a, a+\frac{b-a}{2^{\lfloor z\rfloor}}\right]$

$$G^f(\overline{x}, z, a, b, y) = \begin{cases} 1, & \text{if } h^f\left(\overline{x}, a, a+\frac{b-a}{2^{\lfloor z\rfloor}}\right) = 0, \\ 0, & \text{otherwise} \end{cases}$$

for $y \in \left(a + \frac{(k-1)(b-a)}{2^{\lfloor z\rfloor}}, a + \frac{k(b-a)}{2^{\lfloor z\rfloor}}\right)$ (where $k = 2, 3, \ldots, 2^n$) we have:

$$G^f(\overline{x}, z, a, b, y) = \begin{cases} 1, & \text{if } \prod_{i=1}^{k-1} h^f\left(\overline{x}, a+\frac{(i-1)(b-a)}{2^{\lfloor z\rfloor}}, a+\frac{i(b-a)}{2^{\lfloor z\rfloor}}\right) \neq 0 \\ & \wedge h^f\left(\overline{x}, a+\frac{(k-1)(b-a)}{2^{\lfloor z\rfloor}}, a+\frac{k(b-a)}{2^{\lfloor z\rfloor}}\right) = 0, \\ 0, & \text{otherwise} \end{cases}$$

and for $Y \notin [A, B]$ the function g_x^f is equal to 2.

The definition of G_f is given by the cases with respect to the value of the expression given by $\prod h^f$, since for $f \in L_m$, the function $h_f \in L_{m+p+2}$ and $G^f \in L_{m+2p+3}$. Then we have $K_m^f \in L_{m+2p+6}$. Now we must use the function K_m^f in the same way as K^f which gives us Z_f in the class L_{m+3p+9}. The final definition of $g(\overline{x}) = m_y f(\overline{x}, y)$ ([5] Theorem 4.3) given below

$$g(\overline{x}) = \begin{cases} Z^{f^+}(\overline{x},0) - Z^{f^-}(\overline{x},0), & \text{if } S^{f^+}(\overline{x}) < \frac{1}{12} \wedge S^{f^-}(\overline{x}) < \frac{1}{12}, \\ Z^{f^+}(\overline{x},0), & \text{if } \left(S^{f^+}(\overline{x}) \geq \frac{1}{12} \wedge S^{f^-}(\overline{x}) < \frac{1}{12} \right) \\ & \text{or} \\ & \left(S^{f^+}(\overline{x}) < \frac{1}{12} \wedge S^{f^-}(\overline{x}) < \frac{1}{12} \right. \\ & \left. \wedge Z^{f^+}(\overline{x},0) < Z^{f^-}(\overline{x},0) \right), \\ -Z^{f^-}(\overline{x},0), & \text{if } \left(S^{f^+}(\overline{x}) < \frac{1}{12} \wedge S^{f^-}(\overline{x}) \geq \frac{1}{12} \right) \\ & \text{or} \\ & \left(S^{f^+}(\overline{x}) < \frac{1}{12} \wedge S^{f^-}(\overline{x}) < \frac{1}{12} \right. \\ & \left. \wedge Z^{f^+}(\overline{x},0) \geq Z^{f^-}(\overline{x},0) \right), \end{cases}$$

where $f^+(\overline{x},y) = \begin{cases} f(\overline{x},y), & y \geq 0, \\ 1, & y < 0; \end{cases}$ $f^-(\overline{x},y) = \begin{cases} f(\overline{x},-y), & y > 0, \\ 1, & y \leq 0; \end{cases}$ remains the class of g identical to the class of Z^f, i.e. $g \in L_{m+3p+9}$.□

Theorem 4.3 *Let* $f : R^n \to R$ *be an* ***R****-recursive function. Then for all* $i \geq 0$ *if* $f \in M_i$ *then* $f \in L_{(3p+9)i}$.

The above statement is a simple consequence of the fact $M_0 = L_0$ and Lemma 4.2.

5. Conclusions

In the paper we give the first rough approximation of 'a complexity' of limit operations in the terms of the *m*-operator and conversely. The results, interpreted in the intuitional way, can suggest what kind of connection exists between infinite limits and a *m*-operator.

We also establish the proper relation between the levels of the limit hierarchy and *m*-hierarchy. Let us point out that in consequence we may investigate analogies which exist for the limit hierarchy (also *m*-hierarchy) and Baire classes

[7]. Also the kind of a connection between the $\sum_n^0 -$measurable functions and ***R***-recursive functions is an open problem.

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