On the existence of connections with a prescribed skew-symmetric Ricci tensor

Abstract. We study the so-called inverse problem. Namely, given a prescribed skew-symmetric Ricci tensor we find (locally) a respective linear connection.

1. Introduction. All manifolds and maps between manifolds considered in the paper are assumed to be smooth (i.e. of class $C^\infty$).

The concept of a linear connection $\nabla$ on a manifold $M$ and its Ricci tensor $S$ can be found in the fundamental monograph [4].

In the present paper, we study the so-called inverse problem.

More detailed, under some assumption on a tensor field $r$ of type $(0, 2)$ on $M$, we prove the existence of a local solution of the equation

$$S = r$$

with unknown linear connection $\nabla$ on $M$.

In particular, we deduce that any 2-form $\omega$ on a manifold $M$ with $\dim(M) \geq 2$ is locally the Ricci tensor $S$ of some linear connection $\nabla$ on $M$.

In the analytic situation, the inverse problem was studied in many papers, e.g. [1, 2, 3, 5]. For example, in [5], using the Cauchy–Kowalevski theorem, the authors found (locally) all analytic linear connections for a prescribed analytic Ricci tensor. In the $C^\infty$ situation, we can not apply the Cauchy–Kowalevski theorem.

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From now on, \( x^1, \ldots, x^n \) denote the usual coordinates on \( \mathbb{R}^n \) and \( \partial_1, \ldots, \partial_n \) denote the usual canonical vector fields on \( \mathbb{R}^n \). Given a map \( f : \mathbb{R}^n \to \mathbb{R} \) let \( (f)_i := \partial_i(f) = \frac{\partial f}{\partial x^i} \) for \( i = 1, \ldots, n \).

2. The main result. The main result of the paper is the following

**Theorem 1.** Let \( M \) be a manifold such that \( \dim(M) \geq 2 \) and let \( x_0 \in M \). Let \( r \) be a tensor field of type \((0,2)\) on \( M \) such that \( r(X,X) = 0 \) around \( x_0 \) for some vector field \( X \in X(M) \) with \( X_{x_0} \neq 0 \). Then there is a linear connection \( \nabla \) on \( M \) such that \( r \) is the Ricci tensor \( S \) of \( \nabla \) on some neighborhood of \( x_0 \).

**Proof.** We may assume that \( M = \mathbb{R}^n \), \( x_0 = 0 \) and \( X = \partial_1 \).

Let \( r \) be the tensor field of type \((0,2)\) on \( \mathbb{R}^n \) and denote \( r_{ij} = r(\partial_i, \partial_j) \) for \( i, j = 1, \ldots, n \). Then
\[
(2) \quad r_{11} = 0.
\]

The Ricci tensor \( S \) of a linear connection \( \nabla \) has the following rather well-known coordinate expression
\[
(3) \quad S(\partial_i, \partial_j) = \sum_{k=1}^{n} \left[ (\Gamma^k_{ij})_k - (\Gamma^k_{kj})_i \right] + \sum_{k,l=1}^{n} \left[ \Gamma^i_{lj} \Gamma^k_{kl} - \Gamma^i_{kj} \Gamma^k_{il} \right], \quad i, j = 1, \ldots, n,
\]
where \( \Gamma^i_{jk} \) are the Christoffel symbols of \( \nabla \), see [4].

It is sufficient to show that under assumption (2), equation (1) has a local solution (defined on some neighborhood of \( 0 \)) \( \nabla = (\Gamma^a_{bc}) \) such that
\[
(4) \quad \Gamma^a_{bc} = 0 \text{ for } a = 3, \ldots, n, \quad b, c = 1, \ldots, n,
\]
\[
\Gamma^2_{bc} = 0 \text{ for } b, c = 2, \ldots, n,
\]
\[
\Gamma^1_{bb} = 0 \text{ for } b = 1, \ldots, n,
\]
\[
\Gamma^1_{1b} = 0 \text{ for } b = 1, \ldots, n.
\]

In other words, we put \( \Gamma^a_{bc} = 0 \) for \( a, b, c = 1, \ldots, n \) except for \( \Gamma^2_{ij} \) with \( j = 2, \ldots, n \) and \( \Gamma^1_{ij} \) with \( i = 2, \ldots, n \) and \( j = 1, \ldots, n \).

Using (4) and the coordinate expression (3), we get
\[
(5) \quad S(\partial_i, \partial_j) = \sum_{k=1}^{2} \left( (\Gamma^k_{ij})_k - (\Gamma^k_{kj})_i \right) + \sum_{k,l \in \{1,2\}, k \neq l} \Gamma^1_{kj} \Gamma^k_{il} = (\Gamma^1_{ij})_1 + (\Gamma^2_{ij})_2 - \Gamma^1_{2j} \Gamma^1_{i1} - \Gamma^2_{1j} \Gamma^1_{i2}
\]
as \( \Gamma^a_{bc} = 0 \) if \( a = 3, \ldots, n \) and \( b, c = 1, \ldots, n \), and \( \Gamma^a_{ac} = 0 \) if \( a, c = 1, \ldots, n \).

Then using (5) and (4), we get
\[
(6) \quad S(\partial_1, \partial_1) = 0,
\]
\[
(7) \quad S(\partial_1, \partial_j) = (\Gamma^2_{1j})_2 \text{ for } j = 2, \ldots, n,
\]
On the existence of connections...

(8) \[ S(\partial_i, \partial_1) = (\Gamma_{i1}^1)_{1} \text{ for } i = 2, \ldots, n, \]

(9) \[ S(\partial_i, \partial_j) = (\Gamma_{ij}^1)_{1} - \Gamma_{ij}^2 \Gamma_{i2}^1 \text{ for } i, j = 2, \ldots, n. \]

More precisely, to obtain (6) we use (5) with \((i, j) = (1, 1)\) and the assumed (in (4)) conditions \(\Gamma_{11}^1 = \Gamma_{11}^2 = 0\). To obtain (7), we use (5) with \((i, j) = (1, 1)\) and the assumed (in (4)) conditions \(\Gamma_{21}^2 = \Gamma_{12}^1 = \Gamma_{1j}^1 = 0\).

Then, by (2), (4) and (6)–(9), the equation (1) with unknown \(\nabla\) satisfying (4) is equivalent to the system of systems of differential equations

(10) \[ (\Gamma_{ij}^2)_{2} = r_{1j} \text{ for } j = 2, \ldots, n, \]

(11) \[ (\Gamma_{i1}^1)_{1} = r_{i1} \text{ for } i = 2, \ldots, n, \]

(12) \[ (\Gamma_{ij}^1)_{1} = \Gamma_{ij}^2 \Gamma_{i2}^1 + r_{ij} \text{ for } i, j = 2, \ldots, n. \]

It remains to observe that the system (10)–(12) has a solution of class \(C^\infty\).

We see that the solution of (10) is

\[ \Gamma_{ij}^2(x) = \int_0^x r_{1j}(x^1, t, x^3, \ldots, x^n)dt + a_j(x^1, x^3, \ldots, x^n) \]

for \(j = 2, \ldots, n\), and that the solution of (11) is

\[ \Gamma_{i1}^1(x) = \int_0^x r_{i1}(t, x^2, \ldots, x^n)dt + b_i(x^2, \ldots, x^n) \]

for \(i = 2, \ldots, n\), where \(a_j, b_i\) are arbitrary maps in \(n - 1\) variables.

Substituting the obtained \(\Gamma_{ij}^2\) into (12), we get the system of ordinary first order differential equations with parameters \(x^2, \ldots, x^n\).

Such obtained system (12) has a solution of class \(C^\infty\) according to the well-known theory of differential equations. We can even solve it explicitly as follows.

Each of the equations

\[ (\Gamma_{i2}^1)_{1} = \Gamma_{i2}^2 \Gamma_{i2}^1 + r_{i2} \text{ for } i = 2, \ldots, n \]

(from the system (12)) is linear non-homogeneous with parameters. Solving them separately (using the well-known method), we obtain

\[ \Gamma_{i2}^1(x^1, \ldots, x^n) = \left( \int_0^x r_{i2}(t, x^2, \ldots, x^n)e^{-\int_0^t \Gamma_{12}^2(\tau, x^2, \ldots, x^n)d\tau}d\tau + c_{i2}(x^2, \ldots, x^n) \right) \times e^{\int_0^x \Gamma_{12}^2(t, x^2, \ldots, x^n)dt} \]
for $i = 2, \ldots, n$, where $c_{i2}$ are arbitrary maps in $n - 1$ variables. Then the other equations of (12) (with $\Gamma_{i2}^1$ as above) have solutions given by

$$\Gamma_{ij}^1(x^1, \ldots, x^n) = \int_0^{x^1} (\Gamma_{ij}^2(t, x^2, \ldots, x^n)\Gamma_{i2}^1(t, x^2, \ldots, x^n) + r_{ij}(t, x^2, \ldots, x^n))\,dt + d_{ij}(x^2, \ldots, x^n),$$

where $d_{ij}$ are arbitrary maps in $n - 1$ variables.

The proof of Theorem 1 is now complete. □

We have the following interesting corollary of Theorem 1.

**Corollary 1.** Let $M$ be a manifold such that $\dim(M) \geq 2$ and let $x_o \in M$. Let $\omega$ be a 2-form on $M$. Then there is a linear connection $\nabla$ on $M$ such that $\omega$ is the Ricci tensor $S$ of $\nabla$ on some neighborhood of $x_o$.

**Proof.** For any vector field $X$ (in particular with $X_{x_o} \neq 0$) we have $\omega(X, X) = 0$. Then we apply Theorem 1 with $\omega$ playing the role of $r$. □

**References**


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